Executive Summary

This paper will prove how it is possible to quantitively predict business growth, competition, cooperation and M&A etc. through expanding differential equations. In a way, this could be seen as mathematical proof of Michael Porter's theories on competitive strategy, or the cluster theory, but I have written it in a way that real businesspeople can use, by introducing ways to create management vision, analyze competitors and create strategy and business plans, all with examples. The final chapter, "fluctuations" focuses on the cycle of economic conditions, and why macro economic phenomena occur - all shown through mathematical principles.



Introduction

How much is management theory used by executives or business managers in their daily decision making? In the real world, do "theories" and "implementation" go together like the wheels of a car? In the world of physics, theorists prove theories through experimentation and observation, and it is only after a theory's existence is confirmed that it is accepted. For example, Einstein stated that gravity bends light in his theory of relativity, but this was only proven when it was observed that the gravity of the sun bends light. Physics has developed in this way, with theories being created

and tested over and over again. Afterwards, these theories are replaced with mathematical formulas.

The asteroid exploration satellite "Hayabusa" traveled to the asteroid Itokawa, 300 million kilometers away from the Earth. It landed, collected a sample and then returned to the Earth. Newtons Laws of Motion made this possible, and the theories were implemented by various engineering technologies. Path changes and position control were conducted using mathematical formulas outlined in the theories, substituting real values and sending the results to the satellite. The theory and implementation worked together perfectly to make the world's first sample return possible.

But what about management? Generally social science is known as being far more difficult to experiment with compared to natural sciences, but is this true? We need to start by questioning common perceptions like this. This may be hard, but that's no reason not to try. It could be said that theorists must be involved in management themselves and be able to consider what is true and what isn't, before those implementing the theory can learn it and implement it in their daily management activities. If both these roles work together to verify the results of their experiments, management studies will be able to progress.

The author of this paper is currently a manager himself. Generally, all sorts of companies create a budget for each year, and compare results at the middle and end of each financial period. The manager is then judged based on these results. They shouldn't have used too little or too much money compared to the budget. Managers that have to go through this each year must feel that "you can't make a budget with a theory, and nor can you eat with it". This reflects a fundamental difference between physics and management studies. In physics, theories are represented by formulas, but in management, they are represented in words. Formulas allow for entering values (parameters) to give results - making them useful for predicting future movements or phenomenon quantitatively. However management theory does not use formulas, meaning that it is only possible to predict the future qualitatively. This is what naturally leads managers, who's performance is judged based on numbers, to have little interest in theory. "Hayabusa" was able to return to the Earth because the results of entering parameters into a formula (theory) were true, and that these results were effectively transmitted to the satellite. Why can't management be like this too?

What are some examples of formulas that can predict the future? For example, let's say there is some boiling water. If you leave it, it will gradually get cooler, and eventually cool to room temperature. We all can predict this as we have experienced it. But what if we wanted to know how many minutes it would take to cool to room temperature? We need a formula to calculate this. First, we need to create a hypothesis based on the law of cooling. What affects the speed of cooling? Let's try using the difference between the water temperature and room temperature. Let's put this in a formula. If the water temperature is T, and the proportionality constant is k...

Rate of water temperature change =k(T-Room Temperature) $\cdot \cdot \cdot (1)$

This is the law for the cooling of water temperature. Water temperature (*T*) changes along with time (*t*), so this is shown as dT/dt, and if the room temperature were $28^{\circ}C$, the formula for ① would be shown as...

$$dT/dt = k(T-28) \cdot \cdot \cdot (2)$$

Now we have a formula that can be used to calculate. This formula expresses the law that water cools progressively each minute, so it is known as a differential equation. The accumulation of the minutes will lead to a consistent change in water temperature. This accumulation is called integration. We can show the integral of (2) in the following way.

$$T=A \cdot e^{-kt} + 28 \cdot \cdot \cdot \quad ③$$

A represents a fixed number. If the temperature of the hot water was initially 78 degrees i.e. t=0, T=78, so \Im would become...

 $78 = A \cdot e^{-k^*0} + 28 = A + 28$, and A would equal 50. Therefore, \Im would be...

$$T=50 \cdot e^{-kt} + 28 \cdot \cdot \cdot 4$$

(*d*) can be used to show the future water temperature *T*. The author actually measured the water temperatures, which are represented by the blue dotted curve below. The green curve is the one drawn by the formula (*d*) with the chosen constant k = 0.0458.



Water temperature change by time under the room temperature of 28°C.

The hypothesis that "The change in water temperature is proportional to the difference between the room temperature and water temperature" has been proven as correct. In fact, the time it takes for hot water of 78 degrees to cool to room temperature of 28 degrees was found by the formula to be 100 minutes, and in actual fact it was 104 minutes.

In management, the word "differentiation" is often used. In other words, "doing something different from the competition". So how can we represent "different from the competition" in a formula? A hint comes from biology. In biology, various species fight for survival. They might even fight for the same food source. The more similar the type and size of the species, the closer the fight will be. However, if one species has a different type of food source and lives in a different place to other species, that species can live without being significantly affected by others. In biology, there is a formula to show quantitatively how much a species must differ from others for it to survive.

By using this formula, we might be able to give a mathematical formula to business differentiation, and use it to quantitively evaluate. There is a field of biology called "Mathematical Biology", which uses differential equations to model and explain breeding and the spread of contagious diseases. Similar to physics, a hypothesis is created followed by a differential equation, and the results of this are compared to the real data. Particularly, the famous *Lotka-Volterra model* is useful as it is both simple and explains the movement of different organisms well.

In *Chapter 1*, the differential equation model for the increase of organism numbers will be applied to management, and it will be used to explain a way to predict your company's growth. *Chapter 2* will focus on competition. Your company affects your competitors, and you are affected by your competitors. You will find that the *Lotka-Volterra competition model* can be used to explain this well. In *Chapter 3*, you will learn how to find the parameters that will be the keys to the competition model, before actually predicting the future and comparing the results to real life performance. In *Chapter 4*, the cooperation model, which is created by reversing the positive and negative numbers in the competition coefficient, will be discussed. In *Chapter 5*, competitive relationships that arise from adding time fluctuations (for more natural results) to each coefficient will be discussed. In the sixth and final chapter, the problems with the theory, and the possibilities will be mentioned.

Chapter 1

Applying differential equations to management

Let's define x as the population of an organism, and r as the internal growth rate (difference between birth and death rates). In this case, the rate of change in the population of an organism in the moment t can be expressed with the differential equation below.

$$dx/dt = rx \quad (1.1)$$

This population theory was proposed by the British economist Malthus. If the initial population is defined as x_0 . The solution to the above equation (by integrating the formula (1.1)) would be as shown below.

 $x = x_0 e^{rt}$ (1.2)

We can substitute $x_0 = 10$, r = 0.1 into (1.2) and increase the time, **t**, to get a result as shown in the graph below, where **x** grows into a bigger and bigger number.



At this rate, there will be no limit to the rapid rise in population, so a change must be made. This is because as the population of a certain organism increases, the individual organisms must scramble for food, and will have increasingly limited space to live. The excrement of each organism will pollute the environment, and this will quickly lead to disease due to the high population density. Therefore, there is a need to limit the increase at a certain point in time. We can represent this "limit" mathematically with "rate of change = 0". In other words, there is a need for the first equation (1.1), which represent the rate of change, to become zero at some point. Let's say K represents that point in time - we can rewrite equation (1.1) so that when x meets K, the rate of change becomes zero. The Dutch mathematical biologist Verhulst proposed the below formula.

dx/dt = rx(1-x/K) (1.3)

He has multiplied Malthus' equation by (1-x/K). According to this equation, when **x** reaches **K**, (1-x/K) = 0 and the change stops. If you look at it differently, x/K is the ratio that shows how much of an organism's *carrying capacity* has been reached, and (1-x/K) is the energy left in the environment in which the organism will grow. As **x**

increases, the capacity of the environment to take in new organisms decreases, and eventually becomes zero.

The below equation (1.4) was created by integrating (1.3) and changing **x** into a function of time.

$x=K/(1+(K/x_0-1)e^{-rt})$ (1.4)

Figure 2 is a comparison of a US population growth prediction made using this formula, and the actual population growth. Up until the 1930s, the formula is extraordinarily accurate. From the 1940s, it becomes less accurate due to the fact that Verhulst set the maximum for population at 197,000,000. (the US population is already higher than 300 million, and Verhulst chose the growth rate r = 0.3134 with a starting value of 3.9 million) It is therefore surprising that the differential equation (1.3) was able to accurately predict population growth up to 100 years in the future.



Figure 2

The Verhulst prediction (red line) is known as the *S*-curve or *Logistic Curve* from its shape.

But do similar phenomena occur in the business world? Let's take a look. *Table 1* shows the changes in sales of a Japanese intermediate bulk container (IBC) rental business. *Figure 3* shows the information on a scatter plot. To see if this is the same as the logistic growth curve, we need to find the below values (parameters) from

equation (1.4).

Growth rate: **r** Charring capacity: **K** Initial value: **x**₀

We don't know the growth rate at this point, so let's use the growth rate that Verhulst used when he predicted US population growth, I.e. r=0.3. For the carrying capacity K, this can be seen as the maximum amount of sales that the business can make. The overall market value for IBC is about JPY8 billion and the market for IBCs with inner bags (liners) is about 20% of that. If our share was estimated at 75% of that market, we could find K as below.

K = JPY 8 billion x 20% x 75% = JPY 1.2 billion

The *initial value* x_0 can be the sales value for the first year, 17 million yen. Now we have all three necessary parameters.

Table 1

	Years Passed	Sales Revenue (JPY million)
1994	0	17
1995	1	58
1996	2	85
1997	3	146
1998	4	230
1999	5	265
2000	6	379
2001	7	563
2002	8	687
2003	9	751
2004	10	870
2005	11	986
2006	12	1,083
2007	13	1,110
2008	14	1,170
2009	15	1,098
2010	16	1,122
2011	17	1,190



Figure 3-2







<Step 1> Use the sales numbers from *Table 1* as the vertical axis and the years passed as the horizontal axis to create a scatter plot. (*Figure 3-1*)

 $\langle Step 2 \rangle$ Draw the logistic growth curve for *r*=0.3, *K*=12.0, *x*₀=0.17 (*Figure 3-2*) $\langle Step 3 \rangle$ Change the growth rate r to have the logistic growth curve be close to the actual value (*Figure 3-3*)

In $\langle Step 2 \rangle$, the slope of the logistics growth curve showed a more gentle increase than the actual growth, so we can see that the growth of this business happened at a faster rate than that of US population growth. If we gradually increase *r* to the point where the logistics growth curve is closest to the actual growth rate, we can see that this business has a growth rate of about 0.55. You can see that the actual growth rate and the calculated logistic growth curve overlap pretty accurately.

However, you might say that it was just a coincidence that the actual and predicted growth rates were similar with this example. The Japanese cabinet office conducts consumer trends surveys each year and publicizes information on consumer goods consumption rates, so let's use that information to test the theory further. *Figures 4-1* and *4-2* were created by applying the above <Steps 1-3> to the information on consumer products. Each figure shows the calculated logistics growth curve as a dotted line - you can tell how close the predicted growth is to the actual number. *Table 2* shows the actual growth rates in addition to other information.



Figure 4-1

Source: Japanese Cabinet Office consumer trends survey

Figure 4-2



Japanese nominal GDP changes (Wikipedia)

Table 2

List of growth rates (reference values)

# of stores	r
Convenience Store	0.22
Hamburger Store	0.35
Hair Salon	0.33

Organism growth/multiplication	r
Average height (male)	0.30
Multiplication of aerogenes bacteria	1.10

Adoption rates of consumer durables	r
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	Company sales revenues	r
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Electric washing machine	0.30
Electric refrigerator	0.45
Piano	0.16
Electric vacuum	0.33
Car	0.25
Stereo	0.35
Color TV	1.08
Air conditioner	0.25
Video camera	0.30
VTR	0.45
Clothes dryer	0.20
PC	0.30
Flat screen TV	0.60
DVD	0.35
Mobile phone	0.70
Average	0.40

SoftBank (2003≁)	0.90
NTT Docomo (1994~)	0.75
Sony (1989~)	0.35

Real GDP	r
Japan (1955~)	0.10

The average growth rate for durable consumer goods is 0.40. Goods like color televisions (1.08) and mobile phones (0.70) that are used in daily life have a higher growth rate, while things like pianos (0.16) and bicycles (0.25) that are not essential for daily life have lower growth rates. The Japanese real GDP growth rate is close to 0.10. Through this experiment it has become clear that the logistic growth curve can be used to accurately predict the growth of a great variety of things. Let's move on to how we can make use of these tools in management. As with the experiments above, the logistic growth curve is decided based on just three coefficients i.e. Growth rate (r), carrying capacity (K) and initial value (x_0)). Therefore, when using the concept for business, these values will decide on the future of the business, and it is essential to take great care in setting their value.

<Growth coefficient "r" >

The growth coefficient r is known in biology as the intrinsic rate of natural increase. (it will be called the *growth coefficient* in this paper) As the value of r increases, so does the speed of growth, so if the x axis of a graph represents time, the logistic curve will have a more sudden incline. On the other hand, if the r value is smaller, the growth rate will decrease and the curve will have a more gentle incline. Depending on what data you have, it is common to not have data for every year from the initial value, and it may be difficult to determine what data should be used as the value for r. Therefore it is realistic to use the shooting method that we used previously.

When a relatively long time has passed since the business started, and there is sufficient data on performance;

- ① Create a scatter plot with sales revenues on the vertical axis and time on the horizontal axis.
- (2) Calculate the maximum market share that your company can achieve based on the size of the overall market and use it to define the *K* value.
- (3) Change the r value of the logistic growth curve until the curve is close to the actual performance curve. This r value becomes the growth rate. (Shown in *Figures 3-1 to 3-3*)

When there is no (or only a little) performance data;

- (4) Decide on the r value by using the industry leader (or another existing company's) data and applying <Step 1> above.
- (5) If you can't source any data (or if there are no similar businesses or products), use the *r* list (*Table 2*) as a reference and use the *r* value for products in a similar industry.
- (6) If this is difficult, use r = 0.4-0.5 and use the fshooting method later to edit the value.

<Carrying capacity "K">

The reason that Verhulst's prediction of US population growth (*Figure 2*) was only correct up to the 1930s was because he did not consider more than 100 years into the future. From this, you can see how the predictions are largely affected by the K value. So what is the K value in business? When considering markets, products and services, there are things that we can see, and future potentials that we can't yet see. *Figure 5* illustrates this.

Figure 5

		Product/		Service
		Current		Potential
Market <i>Potential</i>	Current	K 1	K2	Κ.
		K2	K 2	₹4
	ŀ	€ 3	K₅	

From the point of view of management

- K1 Management focused on maintaining current state
- K₂ Management focused on market share
- K₃ Management focused on growth, considering market potential
- K4 Management focused on growth, considering the growth of a product or service
- K₅ Management focused on creating a new market with a new product or service



Figure 6 represents the information from *Figure* 5 on the carrying capacity curve (normal distribution). In order to set a value for K, we first need to get hold of the current market size(K_2). Next we can make K bigger by adding an international market if the domestic market is too saturated (K_3), or if one particular product or service has reached maturity, we can add another product or servic K_2 o the equation (K_4) to expand market scope, or take other measures to create new potential markets (K_5). As shown in *Chapter* 1, once K is reached, the growth of the business will stop. It is possible to decide on a different value for K for short, mid and long term goals, but at least for the long term value, K should represent the ultimate goal that management wants to reach. (therefore, this is also a process that allows one to find out what management is focused on achieving)

Management that is focused on maintaining its current position (K_1) would be able to

define *K* as its current sales volume. As their goal is solely to maintain their current position, they would not be interested in the market size or their market share, and would not be that sensitive to the actions of other companies in their industry. K_2 management style is similar to this, but is more aware of the market size and their company's market share. K_3 and K_4 management are focused on company growth and are interested in potential new markets and products/services. K_5 management goes as far as creating a new market. To quote the Pareto Principle, 80% of the world's management falls under K_1 and 80% of the remaining 20% (20% x 80% = 16%) are K_2 types, 80% of the remaining 4%, or 4% x 80% = 3.2% are K_3 or K_4 types and the final remaining 0.8% are K_5 .

<Initial Value "x₀" >

The initial value should be the current sales volume if the business already exists, or zero if it is a new business. As seen in *Chapter 2*, as competition becomes fiercer, the initial value will have a bigger impact on the future, so it is best to research the sales composition of other companies in the same industry to ensure that you are making an apples to apples comparison.

(Caution) If using Excel to calculate, you cannot set the starting value as zero, so choose a small real number instead.

<Applying the logistic growth curve to management>

Now we know how to determine the above three coefficients, let's take a closer look at how we can apply the logistic growth curve to management. We will mainly be discussing the below three points.

- 1) Using it for business (budget) plans
- 2) Using it for the creation of a business vision and strategy

1) Using it for business (budget) plans

The logistic growth curve has point symmetry on both sides of the inflection point. The inflection point is the point where the value of the differential equation (1.3) switches between positive and negative - in other words when the speed of growth stops increasing and starts to decrease. The curve can be separated into two parts with this point at the center, and we can further separate these parts into the introduction period and early growth period, and late growth period and plateau period to create a total of four stages. See *Figure 7*.





The logistic growth curve can be separated into the introduction period, early growth period, late growth period and plateau period.

<Introduction Period>

When creating a budget for the starting of a new business or project, or the implementation of a new product, it is common to use a straight line to illustrate expected sales growth. For new businesses, they often have aims like "reach breakeven point in the third year and reduce the accumulated losses to zero in the fifth year", but as you can see with the logistic growth curve, the introduction period is a period of slow growth, and not being able to meet the budget or goals during this period often starts discussions to prematurely remove the products from sale. Break even analyses also use straight lines, and therefore the breakeven point in the budget tends to be set much earlier.

If we refer to budgets created with straight lines as "linear budgets", and budgets created with the logistic growth curve as "non-linear budgets", we can see that the breakeven point on linear budgets (*Figure 8-1*) and non-linear (*Figure 8-2*) budgets differs.

Figure 8-1

Figure 8-2



<Linear budget>

<Non-linear budget>

<Growth Period>

There is a need to separate the growth period at the inflection point into the first and second halves. In the first half, the growth speed keeps increasing, and so will the sales. However, in the second half, the growth speed begins to decrease. Generally, the trends are as outlined below.

Early growth period	Linear budget < Non-linear budget
Late growth period	Linear budget>Non-linear budget

If you create a linear budget for the late growth period based on the early growth period, the expected performance will not be met. Let's take a closer look.

1) Researching examples of "early growth periods"

Take eco-friendly cars, such as hybrids and electric cars, that are considered to be at the early stage of growth. Let's try using the logistic growth curve to predict their future growth. In Japan, hybrid cars have been on the market since 2001. Since then, their sales numbers have gradually increased, and in 2010 they recorded sales of around 0.5 million cars. If we consider the total yearly sales of passenger cars in Japan today as over 3 million cars, and propose that the eco-friendly car market, including electric cars, will eventually reach a 70% market share, the carrying capacity K should be set

at over 2 million cars. The growth coefficient r can be calculated from previous car sales growth as 0.25. However, this number is from a time when people did not have cars to start with. We can see that mobile phones, which took over from landline phones, have a growth coefficient r of 0.7, and flat screen TVs such as LCD also have a high coefficient of 0.6. If we consider the recent sales trends of hybrid cars, we can increase the number to 0.4. The initial value x_0 will be the sales volume in 2001 which was 17,000 cars.

If we supplement these values into (1.4), the results are as shown in *Figure 9*. Through this, we can estimate that the number of eco-friendly cars, including electric cars, that will be sold in Japan each year in the 2020s will be over 2,000,000. The US market is about 6 to 7 times bigger than Japan, so we can estimate that in about 10 years, about 13,000,000 eco-friendly cars will be running in developed countries. If each car costs about $1\sim2$ million yen, the market will be about 13-26 trillion yen. We can also determine from the graph that the inflection point should be around 2014.

Figure 9-1

<Prediction of sales numbers of hybrid and electric cars>



 $x_0=17,000, r=0.4, K=2,100,000$



The thesis was translated into English in 2019. It is interesting to assess the above prediction with the actual numbers of sales afterward. See *Figure 9-2*.

Figure 9-2

x₀=17,000, r=0.4, K=2,100,000 Statistics: Japan Automobile Dealers Association

2) Late growth period

The growth rate slows in the late growth period. There is a need to be careful not to create future investment and staffing plans based on the early growth period, as this will lead to fixed costs increasing in the future. *Figure 10* shows the development of a liquid IBC rental business. The blue line is the total number of IBC containers possessed, and the red line is the number that are being rented out. The growth has slowed since around 2005, but the number of containers possessed (newly purchased) has increased anyway, and as a result, the container utilization rate, which used to be 70-80%, has now decreased. The lower utilization raises the fixed costs ratio, which results in profit decrease.

Figure 10



< Plateau Period >

The Plateau Period does not come about simply because the products and the services have come mature in the market, but because growth has stopped (gone flat) for some reason. Firstly, management must be aware that the company has entered a plateau, and when they are, seeing it as a positive or negative thing is dependent on manager's perspective. Even if there are enough management resources, management that is not focused on growth will make it difficult to bounce back from a plateau period. Management that is aiming for more growth will always be able to consider ways of restarting growth. Let's take a look at an example.

Figure 11 shows the sales trends of a pallet cleaning company. In the short term, the carrying capacity K for the business was decided by the capacity of their cleaning machines, which was reached in 4-5 years, and after that the growth plateaued for over 10 years. It indicates that the manager's vision equals to the current capacity of the cleaning machines. Once the management changed, and they invested in the

latest new machines, they were able to restart growth. As you can see from this example, K is largely affected by the perspective and focus of management. From the graph we can also see that they are reaching their next plateau.

Figure 11





Green : r=1.5, K=5 million yen (current machine) Purple : r=1.3, K=20 million yen (current + new machine)

Figure 12 shows the sales trends of an IBC rental business. The red solid line represents the actual performance, and it is clear that this business has already entered a plateau period. Since it is a market leader domestically, it must expand overseas in order to continue growing (The US market, for example, is 5 to 6 times bigger than Japan). The graph compares predicted growth if it were to expand globally after entering a plateau period (purple line) with predicted growth if it were to start preparing to expand globally during the late growth period (dotted line). We can

see that preparing during the late growth period will result in the first plateau arriving 7-8 years earlier.



<Expansion of an IBC rental business from the domestic to global market>



Green : r=0.6, K=12 million yen (current machine) Purple : r=0.3, K=48 million yen (current + new machine)

2) Applying this concept to creating vision and strategy

Just as organisms grow naturally without realizing it, companies (businesses) must also grow sustainably. In order to achieve this, there must be a leader that defines the direction in which the business will grow, and this leader must create a vision and detailed strategy to achieve it. They must also create detailed sales and profit goals and constantly be comparing them with actual performance. *The logistic growth curve* (*LGC*) is a very strong tool to use when predicting the future of your company. If you use this tool effectively, you can create more accurate and timely measures at any stage of growth. It was discussed earlier on, but below is a quick overview of the reasons the *LGC* is beneficial for management.

You can create a habit of representing your market size numerically

There are a lot of K_1 managers that run their businesses without considering market size. Not being aware of market size means that it is unclear where management is

focused, and that they are simply maintaining the company's current position. Just as organisms are always multiplying, ensuring that a business grows sustainably should be the mission of business management and a responsibility to their stakeholders.

You can make a habit of understanding your current market position and defining your market share goals.

If you do a logistic analysis, you can see that the way that K is decided can have a significant impact on your company's future. Defining K, relative to the market size, reflects where management is focused. Even if you have potential management resources, a company will never grow further than where management has their vision set. Companies that enjoy sustainable growth have management that is constantly setting high goals.

You can learn the average growth rate of the industry in advance

By comparing your company's growth rate with the industry standard growth rate, you can objectively evaluate your performance. This will allow you to avoid taking premature actions such as withdrawing from the market.

You can create a realistic budget

This method allows you to both see what stage of growth your company is currently in, predict future growth and create realistic budgets (and business plans).

You can plan for appropriate investment and staffing based on realistic future growth predictions

Excessive (or inadequate) investment or staffing is caused by a lack of balance with sales performance. A logistic analysis can aid in creating a realistic and effective investment and staffing plan to appropriately control fixed costs.

You can prepare to make the next changes before reaching the carrying capacity K.

This allows you to recreate a vision and strategy in a timely fashion. You will become aware of the time when management must change their perspective and aim higher to ensure continued growth.

The end of Chapter 1 -

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